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DISCRETE MATHEMATICS AND GRAPH THEORY

For

COMPUTER SCIENCE

DISCRETE MATHEMATICS AND GRAPH THEORY

SYLLABUS

Mathematical Logic: Propositional Logic; First Order Logic.
Set Theory & Algebra: Sets; Relations; Functions; Groups; Partial Orders; Lattice; Boolean Algebra.
Combinatory: Permutations; Combinations; Counting; Summation; generating functions; recurrence relations; asymptotic.
Graph Theory: Connectivity; spanning trees; Cut vertices & edges; covering; matching; independent sets; Coloring; Planarity; Isomorphism.

ANALYSIS OF GATE PAPERS

Exam Year	1 Mark Ques.	2 Mark Ques.	Total
2003	5	15	35
2004	5	11	27
2005	5	10	25
2006	3	10	23
2007	4	9	22
2008	4	10	24
2009	4	6	16
2010	6	8	24
2011	-	5	10
2012	6	5	16
2013	6	3	12
2014 Set-1	6	9	24
2014 Set-2	5	8	21
2014 Set-3	7	8	23
2015 Set-1	5	8	21
2015 Set-2	5	8	21
2015 Set-3	7	6	19
2016 Set-1	5	4	13
2016 Set-2	6	3	12
2017 Set-1	3	1	5
2017 Set-2	3	2	7

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1 PROPOSITIONAL LOGIC

1.1 STATEMENT/ PROPOSITION

A statement (or Proposition) is a declarative sentence that is either true or false, but not both. If the statement is true then we assign a value T to it and if it is false then we assign a value F to it. These values T and F are called the **truth values** of the statement.

PROPOSITIONAL VARIABLES

In logic, it is required to draw conclusions from the given statements. Now, instead of writing the statements repeatedly, it is convenient to denote each of the statements by a unique variable, called propositional variable.

1.2 LOGICAL CONNECTIVES AND COMPOUND STATEMENTS

Statements or propositional variables can be combined by means of logical connectives or operators to form a single statement called compound statements (or compound propositions or molecular statement).

1.2.1 LOGICAL CONNECTIVES

Symbol	**Connective**	**Name**
~ OR ¬	Not	Negation
$\wedge$	And	Conjunction
$\vee$	Or	Disjunction
$\longrightarrow$	Implies or if ….. then	Implication or conditional
$\leftrightarrow$	If and only if	Equivalence or biconditional

Truth table of ~ p

P	~ p
T	F
F	T

Truth table of p $\wedge$ q

p	Q	p $\rightarrow$q
T	T	T
T	F	F
F	T	T
F	F	T

Truth table of p $\vee$ q

P	Q	p $\vee$ q
T	T	T
T	F	F
F	T	F
F	F	F

1.2.2 CONDITIONAL STATEMENT (OR IMPLICATION), CONVERSE AND CONTRA POSITIVE

If p and q are statements, then "if p then q" is a compound statement, denoted as $p \rightarrow q$ and referred as a conditional statements, or implication. The implication $p \rightarrow q$ is false only when p is true and q is false; otherwise, it is always true. In this implication, p is called the **hypothesis** (or antecedent or premise) & q is called the **conclusion** (or consequent).

Truth table of p $\rightarrow$ q

p	q	p $\vee$ q
T	T	T
T	F	T
F	T	T
F	F	F

If $p \rightarrow q$ is an implication, then
The **converse** of $p \rightarrow q$ is the implication $q \rightarrow p$,
The **contrapositive** of $p \rightarrow q$ is the implication $\sim q \rightarrow \sim p$,
And the **inverse** of $p \rightarrow q$ is the implication $\sim p \rightarrow \sim q$.

Truth table of q→p (converse)

p	q	p→q	q→p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Truth Table of ~q→~p (contrapositive)

p	q	~p	~q	~q→~p
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Truth Table of ~p→~q (inverse)

p	q	~p	~q	~p→~q
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Bi-conditional Statement

If p and q are statements, then "p if and only if q" is a compound statement, denoted as $p \leftrightarrow q$ and referred as a bi-conditional statement or an equivalence.
The equivalence $p \leftrightarrow q$ is true only when both p and q are true or when both p and q are false.

Truth Table of p↔q

p	Q	p↔q
T	T	T
T	F	F
F	T	F
F	F	T

1.3 TAUTOLOGIES & CONTRADICTIONS

A compound statement that is always true for all possible truth values of its propositional variables, is called a tautology or valid.
A statement that is neither a tautology nor a contradiction is called a contingency. So, its truth table contains both T and F values at least once in its last column.

1.4 LOGICAL EQUIVALENCE

Two compound statements p and q are said to be logically equivalent or simply equivalent, if $p \leftrightarrow q$ is a tautology. If p is equivalent to q then we write $p \equiv q$.

Operations for Propositions

A) Commutative Properties
i) $p \vee q \equiv q \vee p$
ii) $p \wedge q \equiv q \wedge p$

B) Associative Properties
i) $p \vee (q \vee r) \equiv (p \vee q) \vee r$
ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

C) Distributive Properties
i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

D) Idempotent Properties
i) $p \vee q \equiv p$
ii) $p \wedge q \equiv p$

E) Properties of Negation
i) $\sim(\sim p) \equiv p$ (Double negation law)
ii) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$ (De Morgan's law)
iii) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$ (De Morgan's law)

F) Identity Laws
i) $p \wedge \mathbf{T} \equiv p$
ii) $p \vee \mathbf{F} \equiv p$,
Where **T** denotes any proposition that is always true and **F** denotes any proposition that is always false.

G) Domination Laws
i) $p \vee \mathbf{T} \equiv \mathbf{T}$
ii) $p \wedge \mathbf{F} \equiv \mathbf{F}$

H) Absorption Laws
i) $p \vee (p \wedge q) \equiv p$
ii) $p \wedge (p \vee q) \equiv p$

1.5 PREDICATES & QUANTIFIERS

Consider the statement
"x is a positive integer."
This statement cannot have a truth value unless the value of the variable x is specified. The first part, i.e., the variable x,

is called the subject of the statement, while the second part, i.e., "is a positive integer" – refers to a property that the subject of the statement can have, is called the **predicate.** We can express the above statement by P(x), where P denotes the predicate "is a positive integer" and x is the variable. The statement P(x) is also called a **Propositional function** because once a value has been assigned to x, it becomes a proposition and has a truth value. Propositional functions also occur in computer programs. The logic based upon the analysis of predicates in any statement is called **predicate logic or first order logic.**

1.5.1 QUANTIFIERS

Quantification is an another powerful technique to create a statement from a propositional function. There are two types of quantification, namely, universal quantification and existential quantification. The universal quantification of a predicate P(x) is the statement "P(x) is true for all values of x in the universe of discourse".

The universe of discourse is the domain that specifies the possible values of the variable x. The notation $\forall xP(x)$ denotes the universal quantification of P(x). Here the symbol $\forall$ is called the **universal quantifier.** The statement $\forall xP(x)$ can also be stated as "for every xP(x)" or "for all xP(x)".

The existential quantification of a predicate P(x) is the statement "There exists an element x in the universe of discourse for which P(x) is true".

The notation $\exists xP(x)$ denotes the existential quantification of P(x). Here the symbol $\exists$ is called the **existential quantifier.** The statement $\exists xP(x)$ can also be stated as "there is an x such that P(x)", "there is at least one x such that P(x)", "for some xP(x)", or "there exists an x such that P(x)".

1.5.2 PROPERTIES OF QUANTIFIERS

The negation of a quantified statement changes the quantifier and also negates the given statement as mentioned below:

i) $\sim(\forall xP(x)) \equiv \exists x \sim P(x)$ (De Morgan's Law)

ii) $\sim(\exists xP(x)) \equiv \forall x \sim P(x)$ (De Morgan's Law)

iii) $\exists x(P(x) \rightarrow Q(x)) \equiv \forall xP(x) \rightarrow \exists xQ(x)$

iv) $\exists xP(x) \rightarrow \forall xQ(x) \equiv \forall x(P(x) \rightarrow Q(x))$

v) $\exists x(P(x) \vee Q(x)) \equiv \exists xP(x) \vee \exists xQ(x)$

vi) $\sim(\exists x \sim P(x) \equiv \forall xP(x)$

1.6 SOME BASIC TERMS RELATED TO NORMAL FORM

i) Elementary Product: A product of the variables and their negation is called an elementary product.

ii) Elementary Sum: A sum of the variables and their negation is called an elementary sum.

iii) Factor: A factor of the given elementary sum or product, is a part of it and is itself an elementary sum or product.

iv) Minterms : Let p and q be two propositional variables. All possible formulas which consist of product of p or its negation and product of q or its negation, but should not contain both the variable and its negation in any one of the formula are called minterms of p and q.

v) Maxterm:For given variables, the maxtern consists of sums (disjunctions) in which each variable or its negation, but not both, appears only once. Thus the maxterms are the duals of minterms.

1.6.1 DISJUNCTIVE NORMAL FORM (DNF)

A statement which consists of a sum of elementary products of propositional variables and is equivalent to the given compound statement is called a disjunctive normal form of the given statement. This form is not unique for the given statement.

1.6.2 CONJUNCTIVE NORMAL FORM (CNF)

A statement which consists of a product of elementary sums of propositional variables and is equivalent to the given statement is called a conjunctive normal form of the given statement. This form is not unique for the given statement.

1.6.3 PRINCIPAL DISJUNCTIVE NORMAL FORM (PDNF)

For a given formula, an equivalent statement consisting of disjunctions of the minterms only is known as its principal disjunctive normal form or sum-of-products canonical form. The method for obtaining the principal disjunctive normal form of a given statement using truth table is as follows:
For every truth value T in the truth table of the given statement, select the minterm which also has the value T for the same combination of the truth values of the variables involved in the statement. The sum (disjunction) of these minterms will then be equivalent to the given formula and is also the required principal disjunctive normal form for the given statement.

ALTERNATIVE METHOD TO OBTAIN PDNF

The method for finding the principal disjunctive normal form of a given statement without using truth table is as follows:
First replace the biconditionals and conditional by their equivalent formulas containing only the connectives $\vee$, $\wedge$ and $\sim$. Next, the negations are applied to the variables by using De Morgans' laws followed by distributive laws. Any elementary product which is a contradiction such as $p \wedge \sim p$, is dropped. Minterms are obtained in the disjunction by introducing the missing factors. Identical minterms, if appearing in the disjunctions, must be deleted.

1.6.4 PRINCIPAL CONJUNCTIVE NORMAL FORM (PCNF)

For a given formula, an equivalent statement consisting of conjunctions of the maxterms only is known as its principal conjunctive normal form or product-of-sums canonical form. The method for obtaining the principal conjunctive normal form of a given statement using truth table is as follows:
For every truth value F in the truth table of the given statement, select the maxterm which also has the value F for the same combination of the truth values of the variables involved in the statement. The product (conjunction) of these maxterms will then be equivalent to the given formula and is also the required principal conjunctive normal form for the given statement.

Alternative Method to Obtain PCNF

The method for finding the principal conjunctive normal form of a given statement without using truth table, is similar to the one described previously for the principal disjunctive normal form.

1.6.5 To Obtain PCNF from PDNF and Vice – Versa

If the principal disjunctive (or conjunctive) normal form of a given statement S, containing n variables, is known then the principal disjunctive (or conjunctive) normal form of $\sim$ S will consist of the disjunction (or conjunction) of the remaining minterms (or maxterms) which are not present in the principal disjunctive (or conjunctive) normal form of S. Since $S \equiv \sim (\sim S)$, so we can obtain the principal conjunctive (or disjunctive) normal form S by applying De Morgan's laws to the principal disjunctive (or conjunctive) normal form of ~S.

Example : Find PCNF of a statement S whose PDNF is $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r)$.

Solution

First we obtain the principal disjunctive normal form of ~S, which is the sum (disjunction) of those minterms which are not present in the given PDNF of S. Hence the PDNF of ~S is

$(\sim p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r)$

Thus, the PCNF of S $\equiv$ ~[PDNF of (~S)], i.e.

$\equiv \sim((\sim p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r))$

$\equiv (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (p \vee \sim q \vee r)$

Example: Find PDNF of a statement S whose PCNF is $(p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

Solution

First we find the PCNF of ~S, which is the product (conjunction) of those maxterms which do not appear in the given PCNF of S. Hence the PCNF of ~S is $(p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$

Thus, the PDNF of S $\equiv$ ~[PCNF of (~S)],

i.e. $\equiv \sim((p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee r) \wedge (\sim p \vee \sim q \vee \sim r))$

$\equiv (\sim p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r)$.

1.6.6 ARGUMENTS

An argument is an assertion that a given set of statements (or propositions) $p_1, p_2, \ldots, p_n$, called premises (or hypotheses) implies a certain statement c, called the conclusion. Such an argument is denoted by

$p_1, p_2, \ldots, p_n \Rightarrow c$

So, we can say that c logically follows from $p_1, p_2, \ldots, p_n$.

An argument $p_1, p_2, \ldots, p_n \Rightarrow c$ is said to be valid if c is true whenever all the premises $p_1, p_2, \ldots, p_n$ are true. In other words, the argument $p_1, p_2, \ldots, p_n \Rightarrow c$ is valid if and only if the proposition $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow c$ is a tautology. However, a valid argument does not necessarily lead to a true conclusion.

An argument which is not valid is called invalid or a fallacy.

1.7 VALIDITY OF ARGUMENTS USING TRUTH TABLES

For a given set of premises and a conclusion, it is possible to find whether the conclusion logically follows from the given premises by constructing truth tables. We assert that from a set of premises $\{p_1, p_2 \ldots p_n\}$ a conclusion c follows logically if

$p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow c$

First, for all possible combinations of truth values of the atomic variables, we enter the truth values of $p_1, p_2 \ldots p_n$ and c in the table. Next, we observe the rows in which all $p_1, p_2 \ldots, p_n$ have the value T. If, for every such row, c also has the value T, then (1) holds. Alternatively, we may observe the rows in which c has the value F. If, in every such row, at least one of the values of $p_1, p_2, \ldots, p_n$ is F, then (1) also holds.

1.8 RULES OF INFERENCE

Rule of Inference	**Tautological Form**	**Name**
$\dfrac{p}{\therefore p \vee q}$	$p \rightarrow (p \vee q)$	Addition
$\dfrac{p \wedge q}{\therefore p}$	$(p \wedge q) \rightarrow p$	Simplification
P $\dfrac{p \rightarrow q}{\therefore q}$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens
~q $\dfrac{p \rightarrow q}{\therefore \sim q}$	$[\sim q \wedge (p \rightarrow q)] \rightarrow \sim p$	Modus tollens
$p \rightarrow q$ $\dfrac{p \rightarrow q}{\therefore p \rightarrow r}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism

GATE QUESTIONS

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1 MATHEMATICAL LOGIC

Q.1 What is the converse of the following assertion?
I stay only if you go
a) I stay if you go
b) If I stay then you go
c) If you do not go then I do not stay
d) If I do not stay then you go
[GATE-2001]

Q.2 Consider two well-formed formulas in propositional logic
$F_1 : P \Rightarrow \neg P$
$F_2 : (P \Rightarrow \neg P) \vee (\neg P \Rightarrow P)$
Which of the following statements is correct?
a) F_1 is satisfiable, F_2 is valid
b) F_1 is unsatisfiable, F_2 is satisfiable
c) F_1 is unsatisfiable, F_2 is valid
d) F_1 and F_2 are both satisfiable
[GATE-2001]

Q.3 If X then Y unless Z" is represented by which of the following formulas in propositional logic?
("$\neg$") is negation, "$\wedge$" is conjunction, and "$\rightarrow$" is implication)
a) $(X \wedge \neg Z) \rightarrow Y$ b) $(X \wedge \neg Z) \rightarrow Y$
c) $X \rightarrow (Y \wedge \neg Z)$ d) $(X \rightarrow Y) \wedge \neg Z$
[GATE-2002]

Q.4 Which of the following is a valid first order formula? (Here α and β are first order formulae with x as their only free variable)
a) $(\forall x)[\alpha] \Rightarrow (\forall x)[\beta] \Rightarrow (\forall x)[\alpha \Rightarrow \beta]$
b) $(\forall x)[\alpha] \Rightarrow (\exists x)[\alpha \wedge \beta]$
c) $\forall x)[\alpha \vee \beta] \Rightarrow (\exists x)[\alpha] \Rightarrow (\forall x)[\alpha]$
d) $(\forall x)[\alpha \Rightarrow \beta] \Rightarrow ((\forall x)[\alpha] \Rightarrow (\forall x)[\beta])$
[GATE-2003]

Q.5 Consider the following formula α and its two interpretations I_1 and I_2.
$$\alpha : (\forall x)\left[P_x \Leftrightarrow (\forall y)\left[Q_{xy} \Leftrightarrow \neg Q_{yy}\right]\right]$$
$$\Rightarrow (\forall x)[\neg P_x]$$
I_1 : Domain : the set of natural numbers
P_x = 'x is a prime number'
Q_{xy} = 'y divides x'
I_2 : Same as I_1 except that P_x = 'x is a composite number.'
Which of the following statements is true?
a) I_1 satisfies α, I_2 does not
b) I_2 satisfies α, I_1 does not
c) Neither I_2 nor I_1 satisfies α
d) Both I_1 and I_2 satisfy α.
[GATE-2003]

Q.6 The following resolution rule is used in logic programming:
Derive clause $(P \vee Q)$ from clauses $(P \vee R)$, $(Q \vee \neg R)$
Which of the following statements related to this rule is FALSE?
a) $(P \vee R) \wedge (Q \vee \neg R) \Rightarrow (P \vee Q)$ is logically valid
b) $(P \vee Q) \Rightarrow (P \vee R) \wedge (Q \vee \neg R)$ is logically valid
c) $(P \vee Q)$ is satisfiable if and only if $(P \vee R) \wedge (Q \vee \neg R)$ is satisfiable
d) $(P \vee Q) \Rightarrow$ FALSE if and only if both P and Q are unsatisfiable
[GATE-2003]

Q.7 Identify the correct translation into logical notation of the following assertion. Some boys in the class are taller than all the girls
Note : Taller (x, y) is true if x is taller than y.
a) $(\exists x)(\text{boy}(x) \rightarrow (\forall y)(\text{girl}(y) \wedge \text{taller}(x, y)))$
b) $(\exists x)(\text{boy}(x) \wedge (\forall y)(\text{girl}(y) \wedge \text{taller}(x, y)))$
c) $(\exists x)(\text{boy}(x) \rightarrow (\forall y)(\text{girl}(y) \rightarrow \text{taller}(x, y)))$
d) $(\exists x)(\text{boy}(x) \wedge (\forall y)(\text{girl}(y) \rightarrow \text{taller}(x, y)))$
[GATE-2004]

ANSWER KEY:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
(a)	(a)	(a)	(d)	(d)	(b)	(d)	(a)	(b)	(b)	(d)	(d)	(d)	(d)
15	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
(d)	(a)	(b)	(d)	(b)	(b)	(b)	(a)	(b)	(c)	(d)	(a,d)	(d)	(b)
29	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**
(b)	(d)	(d)	(c)	(a)	(c)	(c)	(a)	11	4	(d)	(a)	(b)	(d)
43													
(d)													

EXPLANATIONS

Q.1 (a)

Let p : I say

q : you go

I stay only if you go

$p \to q$

Converse of $p \to q$ is $q \to p$

i.e., I stay if you go

Q.2 (a)

$F_1 : P \to \sim P$

$\Rightarrow \sim P \vee \sim P$

$\Rightarrow \sim P$

So, F_1 is contingency. Hence, F_1 is satisfiable but not valid.

$F_2 : (P \to \sim P)(\sim P \to P)$

$(\sim P \vee \sim P) \vee (\sim (P) \vee P)$

$(\sim P) \vee (P \vee P)$

$\sim P \vee P$

T

So, F_2 is tautology and therefore valid.

Q.3 (a)

If X then Y unless Z is represented by $(X \wedge \neg Z) \to Y$

Q.4 (d)

$(\forall x)[\alpha \Rightarrow \beta] \Rightarrow ((\forall x)[\alpha] \Rightarrow \forall(x)[\beta])$

is a logical equivalence & therefore, a valid first order formula.

Q.5 (d)

$Q_{yy} =$ "y divides y" is always true

$\therefore Q_{xx} \Leftrightarrow \neg Q_{yy}$ is same as $Q_{xx} \Leftrightarrow$ False Now α becomes

$(\forall x)\left[P(x) \Leftrightarrow (\forall y)\left(Q_{xy} \Leftrightarrow \text{false}\right)\right] \Rightarrow (\forall x)\left[\neg P(x)\right]$

Now consider I_1 : P(x) = "x is a prime number".

α becomes (∀ x x is a prime number if and only if ∀ y (y does not divide x)) ⇒ ∀ x (x is not prime) Since, ∀ y (y does not divide x) is always false (since x divides x always). α now becomes

∀ x (x is a prime number ⇔ false) ⇒ ∀ x (x is not a prime).

Which is true.

Now consider I_2:P(x)= "x is a composite number".

Now α becomes

(∀ x x is a composite number if and only if ∀ y (y does not divide x)) ⇒ ∀ x (x is not a composite number)

By same reasoning used above, α now becomes

(∀ x x is a composite number ⇔ false) ⇒ ∀ x (x is not composite) is also true.

ASSIGNMENT QUESTIONS

Q.1 The recurrence relation capturing the optimal execution time of the Towers of Hanoi problem with n discs is
a) $T(n)=2T(n-2)+2$
b) $T(n)=2T(n-1)+n$
c) $T(n)=2T(n/2)+1$
d) $T(n)=2T(n-1)+1$

Q.2 In a class of 200 students, 125 students passed physics,85 students passed chemistry,65 students passed maths. 50 students passed both physics and chemistry, 35 students passed both physics and maths, 30 students passed both chemistry and maths. 15students passed 3 subjects. How many students have not passed any of the three subjects?
a) 15 b)20
c) 25 d) 35

Q.3 The solution to the recurrence equation $T(2^k)= 3T(2^{k-1})+1$; $T(1)=1$ is:
a) 2^k b) $(3^{k+1}-1)/2$
c) $3^{\log k_2}$ d) $2^{\log k_3}$

Q.4 Maximum number of edges in a n-node undirected graph without self loops is
a) n^2 b) $n(n-1)/2$
c) n-1 d) $(n+1)(n)/2$

Q.5 If all the edge weights of an undirected graph are positive, then any subset of edges that connects all the vertices and has minimum total weight is
a) Hamiltonian cycle b) Grid
c) Hypercube d) Tree

Q.6 What is the size of the smallest MIS (Maximal Independent Set) of a chain of nine nodes?
a) 5 b) 4
c) 3 d) 2

Q.7 Consider an undirected random graph of eight vertices. The Probability that there is an edge between a pair of vertices is ½. What is the expected number of unordered cycles of length three?
a) 1/8 b) 1
c) 7 d) 8

Q.8 What is the chromatic number off an n-vertex simple connected graph which does not contain any odd length cycle? Assume $n\geq 2$
a) 2 b) 3
c) n-1 d) n

Q.9 Let G be a simple connected planner graph with 13 vertices and 19 edges. Then, the number of faces in the planner embedding of the graph is:
a) 6 b) 8
c) 9 d) 13

Q.10 Which of the following statements are TRUE for Undirected Graphs?
P: Number of Odd degree vertices is even
Q: Sum of degrees of all vertices is even.
a) P only b) Q only
c) Both P and Q d)Neither P nor Q

Q.11 Let G= (V, E) be a directed graph where V is the set of Vertices and E the set of Edges. Then which one of the following graphs has the same strongly connected components as G?
a) $G_1=(V,E_1)$ where $E_1=\{(u,v)|(u,v)\notin E\}$
b) $G_2=(V,E_2)$ where $E_2=\{(u,v)|(u,v)\in E\}$
c) $G_3=(V,E_2)$ where $E_3=\{(u,v)|$there is a path of length≤ 2 from u to v in E$\}$
d) $G_4= (V_4, E)$ where V_4 is the set of vertices in G which are not isolated.

EXPLANATIONS

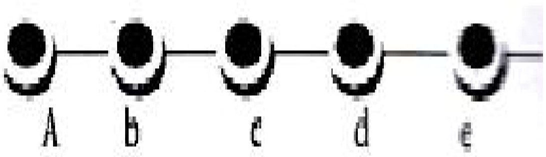
Q.1 **(d)**
Recurrence equation of Towers of Hanoi T(n)=2T(n-1)+1

Q.2 **(c)**
APPLY principle of mutual inclusion and exclusion.
Number of students who passed atleast one of 3 subjects=125+85+65-50-35-30+15 =175
Students have not passed any subject= Total no of students – students who passed atleast one subject
=200-175 =25

Q.3 **(b)**
Let $T(2^k)=X_k$
The recurrence relation reduces to $X_k-3X_{k-1}=1$ where $X_0=1$.
Complementary function= $C_1.3^k$
Particular solution = $\frac{1}{E-3}=\frac{1k}{E-3}=\frac{1k}{1-3}=-\frac{1}{2}$

Q.4 **(b)**
The graph containing maximum number of edges in a n-node undirected graph without selfloops is complete graph.
Each edge corresponds to a pair of distinct vertices. So, Maximum number of edges in complete graph with n-node, k_n is =C(n,2)= $\frac{n(n-1)}{2}$

Q.5 **(d)**



Q.6 **(c)**
The smallest MIS= {b, e, h}

Q.7 **(c)**
We will find the unordered cycle of length 3. So we choose 3 vertices from 8 vertices. This can be done in $8c_3$ ways.
The probability that any three vertices form cycle is 1/2 * 1/2 * 1/2 = 1/8
Expected number of Cycles=∑x P(x)
= $8c_3 * \frac{1}{8}$ = 7

Q.8 **(a)**
In a simple connected graph, if all the cycles are of even length (no cycles of odd length), then it is a bipartite graph. Chromatic number of a bipartite graph is 2. There are no edges within a partition of a bipartite graph and there are only two partitions.

Q.9 **(b)**
By using Euler's Formula

Q.10 **(c)**
By sum of degrees of vertices theorem, both the statements are true.

Q.11 **(b)**
G= (V, E) is directed graph
$G_2=(V,E_2)$ where $E_2=\{(u,v)|(u,v)\in E\}$
G and G_2 have same strongly connected components. Only the difference in G_2 is that all edges of G have been reversed the direction.

Q.12 **(c)**
K_5 is the smallest non-planner graph in terms of number of vertices. So, number of vertices in k_5 is 5 and number of edges in $k_5$is n(n-1)/2 = $\frac{5*4}{2}=10$, where n=5.